# $p$-adic Zeros of Systems of Quadratic Forms

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This survey concerns the following problem. Let $K$ be a field and let $r \in \mathbb{N}$. Define $\beta(r;K)$ to be the largest integer $n$ for which there exist quadratic forms

$$q_i(x_1,\ldots,x_n) \in K[x_1,\ldots,x_n] \quad (1 \le i \le r)$$

having only the trivial common zero over $K$. Thus $\beta(r;K)$ is also the smallest integer such that any such system in at least $1+\beta(r;K)$ variables has a non-trivial common zero. When $r=1$ the number $\beta(1;K)$ is the $u$-invariant of the field $K$.

As examples one has $\beta(1;\mathbb{R}) = \infty$, $\beta(1;\mathbb{C}) = 1$ (since the form $x_1^2$ has only the trivial zero $x_1 = 0$), and indeed $\beta(r;\mathbb{C}) = r$.

We will be primarily interested in the case in which $K$ is a $p$-adic field $\mathbb{Q}_p$. It is well known in this case that $\beta(1;\mathbb{Q}_p) = 4$. For example if $p$ is an odd prime and $k$ is a quadratic non-residue of $p$ then $x_1^2 - kx_2^2 + p(x_3^2 - kx_4^2)$ has only the trivial zero over $\mathbb{Q}_p$, while any form in 5 variables has a non-trivial zero. Thus the key question is what one can say about $\beta(r;\mathbb{Q}_p)$ in general.

Why should one be interested in such problems? Firstly, systems of quadratics are fundamental to Diophantine analysis, since any Diophantine equation may be reduced to such a system. Secondly, in certain circumstances there are local-to-global principles for such systems. For example, suppose we have a system of quadratic forms over $\mathbb{Q}$ which defines a smooth variety, and suppose also that the number of variables $n$ exceeds $2r^2 + 2r$. Then a theorem of Birch [4], proved via the Hardy–Littlewood circle method, shows that there is a non-trivial rational point provided that there is a non-trivial point over every completion of $\mathbb{Q}$. It is clear that one cannot drop the condition over $\mathbb{R}$, but it is natural to ask whether the $p$-adic conditions are satisfied automatically for $n \ge 2r^2 + 2r$. Thirdly, the reduction of general Diophantine equations to systems of quadratics can be made sufficiently efficient in special circumstances that information about values of $\beta(r;\mathbb{Q}_p)$ can yield worthwhile information about higher degree equations. An example of

this occurs in the author's work [11], where it is shown that for any prime $p$ different from 2 or 5, a quartic form over $\mathbb{Q}_p$ in $n$ variables has a nontrivial zero, provided that $n > 16 + \beta(8;\mathbb{Q}_p)$. One would therefore like to know how many variables are needed for a system of 8 quadratic forms to have a non-trivial zero.

What might one expect about $\beta(r;\mathbb{Q}_p)$ ? It was conjectured by Artin, see [2, p.x], that a $p$-adic form of degree $d$ in $n$ variables should have a non-trivial zero as soon as $n > d^2$. A consequence of this would be that a system of $r$ quadratic forms in $n$ variables would have a non-trivial zero as soon as $n > 4r$. On the other hand one may easily construct systems in $4r$ variables having only the trivial zero. Indeed if $q(x_1, x_2, x_3, x_4)$ has only the trivial zero then one may take

$$q_1 = q(x_1, \ldots, x_4),\ q_2 = q(x_5, \ldots, x_8),\ \ldots\ , q_r = q(x_{4r-3}, \ldots, x_{4r}).$$

Thus one is led to the following conjecture.

**Conjecture** *For any $r \in \mathbb{N}$ and any prime number $p$ one has $\beta(r;\mathbb{Q}_p) = 4r$.*

In fact Artin's Conjecture is known to be false (Terjanian [20]), but none of the known counterexamples relate to systems of quadratic forms. Thus the conjecture above remains open.

The most important result known on Artin's Conjecture is probably that of Ax and Kochen [3], who showed that for any given degree $d$ there is a corresponding $p(d)$ such that the conjecture is true for primes $p \geq p(d)$. One may deduce that for any $r$ there is a corresponding $p'(r)$ such that $\beta(r;\mathbb{Q}_p) = 4r$ as soon as $p \geq p'(r)$.

For small integers $r$ more has been proved. As has already been remarked, one has $\beta(1;\mathbb{Q}_p) = 4$ for every prime $p$. This was known implicitly in the 19th century, but was proved explicitly by Hasse [10]. For $r = 2$ one similarly has $\beta(2;\mathbb{Q}_p) = 8$ for every prime $p$, as was established by Demyanov [8] in 1956. However even for $r = 3$ the picture is incomplete. Here it was shown by Schuur [18], building on work of Birch and Lewis [6], that $\beta(3;\mathbb{Q}_p) = 12$ provided that $p \geq 11$. Thus we have the following problem.

**Open Question** *Is it true that $\beta(3;\mathbb{Q}_p) = 12$ for all primes $p$ ?*

There are two major lines of attack on such questions. The first traces its roots through work of Birch, Lewis and Murphy [5] (1962), Birch and Lewis [6] (1965), and Schmidt [17] (1980). The basic idea is to choose representatives $q_i$ for the system in such a way that $q_i(x_1, \ldots, x_n) \in \mathbb{Z}_p[x_1, \ldots, x_n]$. Reduction modulo $p$ then yields forms $Q_i(x_1, \ldots, x_n) \in \mathbb{F}_p[x_1, \ldots, x_n]$, say.

Then, if the system $Q_1,\ldots,Q_r$ has a non-singular zero over $\mathbb{F}_p$ one can lift it to a non-trivial zero of $q_1,\ldots,q_r$ over $\mathbb{Z}_p$, by Hensel's Lemma.

By the Chevalley–Warning Theorem the system $Q_1,\ldots,Q_r$ certainly has a non-trivial zero over $\mathbb{F}_p$ as soon as $n>2r$, so the key issue is whether or not we can produce a non-singular zero. This is clearly not possible in general. Indeed nothing that has been said so far precludes the possibility that the forms $Q_i$ all vanish. Thus the strategy is to start by choosing a good integral model for the system $<q_1,\ldots,q_r>_{\mathbb{Q}_p}$, by removing as many excess factors of $p$ as one can. Here, when we refer to a good model for the system, one should observe that one may make invertible linear changes of variable

$$\underline{\mathrm{x}}\mapsto M\underline{\mathrm{x}},\qquad M\in\mathrm{GL}_n(\mathbb{Q}_p) \tag{1}$$

amongst $x_1,\ldots,x_n$, and invertible linear changes

$$\mathbf{q}\mapsto P\mathbf{q},\qquad P\in\mathrm{GL}_r(\mathbb{Q}_p) \tag{2}$$

amongst the forms $q_1,\ldots,q_r$, without affecting the existence or otherwise of a non-trivial zero. Thus one uses an invariant $\mathcal{I}(q_1,\ldots,q_r)$ of the system (as constructed by Schmidt [17]) which is a function of the various coefficients, and one defines a minimal model to be one in which all the forms $q_i$ are defined over $\mathbb{Z}_p$, and for which the $p$-adic valuation $|\mathcal{I}(q_1,\ldots,q_r)|_p$ is maximal. It can happen that $\mathcal{I}(q_1,\ldots,q_r)=0$, but it is possible to avoid consideration of such systems.

In order to get a feel for what a minimal model might look like, observe that if one takes the transforms (1) and (2) to be $M=pI_n$ and $P=p^{-2}I_r$ respectively, then we return to the original system. Thus we may think of transforms in which

$$|\det(P)|_p=|\det(M)|_p^{-2r/n}$$

as "neutral". However if there is a pair of transforms producing an integral system, but for which $|\det(P)|_p>|\det(M)|_p^{-2r/n}$, then we may regard this as having removed at least one factor $p$ from the system. In fact the condition for a minimal model is precisely that one should have

$$|\det(P)|_p\le|\det(M)|_p^{-2r/n} \tag{3}$$

for any transforms that produce another integral system. We shall say that $q_1,\ldots,q_r$ is "minimized" if it meets this condition.

Under the assumption that $n>4r$ we can draw certain conclusions about the system $Q_1,\ldots,Q_r$ over $\mathbb{F}_p$. For example, if there were any form, $Q_1$, say, such that $Q_1(0,0,x_3,x_4,\ldots,x_n)$ vanishes identically, then the transforms $M=\mathrm{Diag}(p,p,1,1,\ldots,1)$ and $P=\mathrm{Diag}(p^{-1},1,1,\ldots,1)$ would map

$q_1, \ldots, q_r$ to another integral system . However these would violate the condition (3). Thus no form $Q_i$, or more generally no form in the linear system generated by $Q_1, \ldots, Q_r$, can be annihilated by setting two variables to zero. In the same way one can show that one cannot annihilate any $k$ of the forms by setting $2k$ variables to zero. If $Q_1, \ldots, Q_r$ satisfy these conditions we will say that the system is "$\mathbb{F}_p$-minimized". Thus if $q_1, \ldots, q_r$ is minimized, then $Q_1, \ldots, Q_r$ is $\mathbb{F}_p$-minimized. However the converse is not true in general. For example, when $r = 1$, $n = 5$ and $p = 3$, the form $q_1(x_1, \ldots, x_5) = x_1^2 + x_2^2 + x_3x_4 + 9x_5^2$ is not minimized, since we can take $M$ as $\mathrm{Diag}(1, 1, 1, 1, 1/3)$ and $P$ as the $1\times 1$ identity matrix. Then $|\det(P)|_3 = 1$, while $|\det(M)|_3^{-2r/n} = 3^{-2/5}$, contravening the condition (3). On the other hand the reduction to $\mathbb{F}_3$ is $Q_1(x_1, \ldots, x_5) = x_1^2 + x_2^2 + x_3x_4$, which is $\mathbb{F}_3$ minimized.

We can illustrate the use of a minimal model by looking at the case $r = 1$, with $n \geq 5$. For a minimal model, $Q_1$ cannot be annihilated by setting two variables to zero. We proceed to make a linear change of variables so as to represent $Q_1$ using as few variables as possible, and put $Q_1(x_1, \ldots, x_n) = Q^*(y_1, \ldots, y_m)$ accordingly. Then we will have $m \geq 3$, by the minimality condition. The Chevalley–Warning Theorem now produces a non-trivial zero of $Q^*$. Such a zero must be non-singular, since otherwise the form $Q^*$ would be degenerate, contrary to hypothesis. This results in a non-singular zero of $Q_1$, to which Hensel's Lemma may be applied, completing the proof. The reader may care to note that this approach allows all the fields $\mathbb{Q}_p$, including the case $p = 2$, to be handled uniformly.

A similar argument handles the case $r = 2$, for $n \geq 9$ (Demyanov [8] and Birch, Lewis and Murphy [5]). In particular, if $n \geq 9$ and $< Q_1, Q_2 >$ is $\mathbb{F}_p$-minimized, then $Q_1$ and $Q_2$ always have a non-singular common zero. When $r = 3$ and $n \geq 13$ there appear to be numerous special cases to consider. The work of Birch and Lewis [6] and Schuur [18] proves similarly that if $< Q_1, Q_2, Q_3 >$ is $\mathbb{F}_p$-minimized, then there is a non-singular common zero, provided that $p \geq 11$. However the approach is doomed to fail in general, as the following example shows. We take $p = 2$ and examine the forms

$$Q_1(x_1, \ldots, x_{13}) = x_1x_2 + x_3^2 + x_3x_4 + x_4^2,$$

$$Q_2(x_1, \ldots, x_{13}) = x_5x_6 + x_7^2 + x_7x_8 + x_8^2,$$

$$Q_3(x_1, \ldots, x_{13}) = x_1^2 + x_1x_2 + x_2^2 + x_5x_7 + x_6x_8 + x_7^2 + x_8^2$$

over $\mathbb{F}_2$. We claim that any common zero (over $\mathbb{F}_2$) is a singular zero for $Q_1$, and hence is singular for the whole system. To verify this one easily checks that a non-singular zero of $Q_1$ has $x_1^2 + x_1x_2 + x_2^2 = 1$, and that

$x_5x_7 + x_6x_8 + x_7^2 + x_8^2 = 0$ for any zero of $Q_2$. This is enough to show that $Q_3 = 1$ at any point which is both a nonsingular zero of $Q_1$ and a zero of $Q_2$. The claim then follows. One can also verify that the system is $\mathbb{F}_2$-minimized, which requires a case by case analysis. We give a single example, showing that

$$Q_1 + Q_3 = x_1^2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + x_5x_7 + x_6x_8 + x_7^2 + x_8^2$$

and

$$Q_2 + Q_3 = x_1^2 + x_1x_2 + x_2^2 + (x_5 + x_8)(x_6 + x_7)$$

cannot both vanish on a linear space $L \le \mathbb{F}_2^8$ of dimension 4. It will be convenient to work with a basis $\underline{e}_1, \ldots, \underline{e}_8$ of $\mathbb{F}_2^8$, corresponding to the variables $x_1 \ldots, x_8$. If we set $\underline{e}_5' = \underline{e}_5 + \underline{e}_8$ then on the space

$$V :=< \underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{e}_4, \underline{e}_5', \underline{e}_6, \underline{e}_7 >$$

the forms become

$$x_1^2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + x_5x_6 + x_5x_7 + x_5^2 + x_7^2$$

and

$$x_1^2 + x_1x_2 + x_2^2,$$

both of which must vanish on $L' := V \cap L$, which will have dimension at least 3. The second form vanishes only when $x_1 = x_2 = 0$. Hence $L'$ must be contained in $V' :=< \underline{e}_3, \underline{e}_4, \underline{e}_5', \underline{e}_6, \underline{e}_7 >$. On this latter space the first form $Q_1 + Q_3$ reduces to

$$Q := x_3^2 + x_3x_4 + x_4^2 + x_5x_6 + x_5x_7 + x_5^2 + x_7^2.$$

However $Q$ is non-degenerate on $V'$, and hence cannot vanish on a subspace of dimension 3. This contradiction shows that $Q_1 + Q_3$ and $Q_2 + Q_3$ cannot both vanish on $L$.

We therefore see that this particular line of attack cannot prove that $\beta(r; \mathbb{Q}_p) = 4r$ for all $r$ and $p$. However one might consider working modulo $p^2$ or with higher powers, instead of reducing to $\mathbb{F}_p$. If one works only over $\mathbb{F}_p$ the following result seems the most that one can hope for.

**Theorem 1** *(Heath-Brown [12].) For all $r \in \mathbb{N}$ one has $\beta(r; \mathbb{Q}_p) = 4r$ if $p \ge (2r)^r$. Indeed if $K$ is any finite extension of $\mathbb{Q}_p$ with residue field $F_K$, then $\beta(r; K) = 4r$ if $\#F_K \ge (2r)^r$. Moreover an $F_K$-minimized system $Q_1, \ldots, Q_r$ has a non-singular common zero provided that $\#F_K \ge (2r)^r$.*

One should recall that the Ax–Kochen Theorem [3] yields $\beta(r; Q_p) = 4r$ for $p \geq p(r)$, so one might view the result above as merely giving an explicit value for $p(r)$. However when one looks at extensions $K$ of $\mathbb{Q}_p$ there is a more important difference. The Ax–Kochen result implies that $\beta(r; K) = 4r$ if the characteristic of $F_K$ is at least some value $p(r; [K : \mathbb{Q}_p])$. In contrast Theorem 1 has a condition only on the size of $F_K$. Thus it is conceivable that the Ax–Kochen result never applies when the characteristic of $F_K$ is 2, for example.

The overall plan for the proof of Theorem 1 is to give a lower bound for the overall number of zeros of $Q_1, \ldots, Q_r$ over $F_K$, and to compare this with an upper bound for the number of singular zeros. It turns out that to count common zeros it suffices to count zeros of each individual linear combination $Q_a := a_1 Q_1 + \ldots + a_r Q_r$. The number of zeros of $Q_a$ in $F_K^n$ is approximately $(\#F_K)^{n-1}$, and the discrepancy depends (in part) on the rank of $Q_a$. It therefore turns out that the key step in the proof is to give a good upper bound for the number of vectors $a$ in $F_K^r$ for which $Q_a$ has a given rank. This step uses the minimality conditions.

An interesting corollary to Theorem 1 is provided by the following result of Leep [14].

**Theorem 2** *(Leep.) Let $p$ be a prime and let $L = \mathbb{Q}_p(T_1, \ldots, T_k)$. Then $\beta(1; L) = 2^{2+k}$.*

Thus the $u$-invariant of the function field $L$ is $2^{2+k}$. Before this result there had been much work on the case $k = 1$, culminating in a successful treatment for all primes $p \neq 2$, by Parimala and Suresh [16]. Nothing however was known for $k \geq 2$. Now one can even handle pairs of forms, showing that

$$\beta(2; L) = 2^{3+k}.$$

One striking feature of Leep's result is that, in contrast to Theorem 1, there is no restriction on the size of $p$. It is interesting to see how this comes about. Suppose a quadratic form $q(x_1, \ldots, x_n) \in L[x_1, \ldots, x_n]$ is given. We aim to locate a zero of $q$ in which $x_1, \ldots, x_n$ are polynomials in $T_1, \ldots, T_k$ of degree at most $d$ say, by finding suitable values (in $\mathbb{Q}_p$) for the various coefficients $c_1, \ldots, c_N$ say. The conditions these $c_i$ have to satisfy form a system of a large number ($R$ say) of quadratic forms. Here $N$ and $R$ will depend on $d$, but if $n > 2^{2+k}$ we will have $N > 4R$ for large enough $d$. Thus, by Theorem 1, one can find suitable coefficients $c_i$ provided that $p \geq (2R)^R$. The trick now is to use an extension $L^* = K(T_1, \ldots, T_k)$ of $L$ obtained by taking $K$ to be an extension of $\mathbb{Q}_p$ of odd degree having $\#F_K \geq (2R)^R$. Everything now works as before, with the values of $N$ and $R$ unaffected by this change.

The result is that we obtain a non-trivial solution $q(x_1,\ldots,x_n)=0$ in which the $x_i$ are in $L^*$. Finally we appeal to a result of Springer [19], which shows that a quadratic form over a field $L$ of characteristic different from 2 has a non-trivial zero provided that there is a zero over some odd degree extension of $L$. This completes the proof. (Incidentally, although Springer's statement required characteristic different from 2, David Leep points out that one may prove the result without this restriction by essentially the same method.)

We turn now to the second main line of attack on $\beta(r;\mathbb{Q}_p)$. This will provide upper bounds for $\beta(r;\mathbb{Q}_p)$ which are expected in many cases to be sub-optimal. However the method has the advantage of producing results for every prime $p$. The procedure uses induction on $r$, and originates from work of Leep [13] in 1984. One can see that the first line of attack runs into difficulties when the field $\mathbb{F}_p$ is small — it leaves too little room for manœuvre. The second strategy works purely over $\mathbb{Q}_p$ and so encounters no such problems.

The basic idea is as follows. Suppose we are given forms $q_1,\ldots,q_r$ over $\mathbb{Q}_p$. If one can find a linear space $L$ in $\mathbb{P}(\mathbb{Q}_p)^{n-1}$, with projective dimension $\beta(k;\mathbb{Q}_p)$, such that $q_1,\ldots,q_{r-k}$ all vanish identically on $L$, then the remaining $k$ forms $q_{r-k+1},\ldots,q_r$ will have a zero in $L$, by definition of $\beta(k;\mathbb{Q}_p)$. Thus the focus of this technique is on the number $\beta(r;K,m)$, defined as the largest integer $n$ for which there exist quadratic forms $q_1(x_1,\ldots,x_n),\ldots,q_r(x_1,\ldots,x_n)$ over $K$ such that there is no $K$-linear space of projective dimension $m$ on which the forms vanish identically. The argument above shows now that

$$\beta(r;K)\le\beta(r-k;K,\beta(k;K)) \tag{4}$$

for $k<r$, for any field $K$.

One may estimate $\beta(r;K,m)$ via induction on $m$. Suppose our forms vanish on a projective linear space $L$ of dimension $m-1$, spanned by $\underline{e}_0,\ldots,\underline{e}_{m-1}$ say. We wish to find an additional vector $\underline{e}_m=\underline{e}$ to add to this basis. Let $L^*$ be a complementary linear space for $L$ in $\mathbb{P}(K)^{n-1}$, so that $\dim(L^*)=n-m-1$. We will require $[\underline{e}]$ to belong to $L^*$, which will ensure that $\underline{e}_0,\ldots,\underline{e}_{m-1},\underline{e}$ are linearly independent. In order for our forms to vanish on the span of the extended set $\underline{e}_0,\ldots,\underline{e}_{m-1},\underline{e}$ it suffices that

$$q_i(\underline{e}_j,\underline{e})=0,\quad (1\le i\le r,\ 0\le j\le m-1)$$

and

$$q_i(\underline{e})=0,\quad (1\le i\le r)$$

where $q_i(\underline{x},\underline{y})$ is the bilinear form associated to $q_i$. The first set of conditions restricts $\underline{e}$ to a subspace of $L^*$ of codimension at most $rm$, so that a suitable

$\underline{e}$ must exist, provided that $\beta(r;K) < n - m - rm$. It follows that our basis can be extended whenever $n > (r+1)m + \beta(r;K)$, yielding the inductive inequality

$$\beta(r;K,m) \le \max\left\{\beta(r;K,m-1)\,,\,(r+1)m + \beta(r;K)\right\}.$$

We therefore deduce that

$$\beta(r;K,m) \le (r+1)m + \beta(r;K) \tag{5}$$

for all $m$.

One may combine this with (4) to obtain

$$\beta(r;\mathbb{Q}_p) \le \beta(r-2;\mathbb{Q}_p,\beta(2;\mathbb{Q}_p)) = \beta(r-2;\mathbb{Q}_p,8) \le 8(r-1) + \beta(r-2;\mathbb{Q}_p).$$

Starting from $\beta(1;\mathbb{Q}_p) = 4$ and $\beta(2;\mathbb{Q}_p) = 8$ one then finds that

$$\beta(r;\mathbb{Q}_p) \le \left\{\begin{array}{cc} 2r^2, & r \text{ even}, \\ 2r^2+2, & r \text{ odd}, \end{array}\right. \tag{6}$$

(Martin [15], improving slightly on the original result of Leep). In particular one has

$$\beta(3;\mathbb{Q}_p) \le 20,$$

for all primes $p$.

One may ask whether one can improve on the bound (5). In the case $r = 1$ the estimate (5) becomes $\beta(1;\mathbb{Q}_p,m) \le 2m+4$, and indeed this is best possible. However for $r = 2$ one has only $\beta(2;\mathbb{Q}_p,m) \le 3m+8$, and here one can do better by an argument due to Dietmann [9] (improved slightly by Heath-Brown [11]). The method is based on the following theorem of Amer [1, Satz 8, p.29] in an unpublished thesis.

**Theorem 3** *(Amer, 1976) For any field $K$ of characteristic $\chi_K \ne 2$ one has $\beta(2;K,m) \le \beta(1;K(T),m)$ for every integer $m \ge 0$.*

The special case $m = 0$ is given by Brumer [7]. In an unpublished manuscript Leep shows that the result holds even when $\chi_K = 2$.

In view of (5) one has $\beta(1;K(T),m) \le 2m + \beta(1;K(T))$, so that

$$\beta(2;K,m) \le 2m + \beta(1;K(T)).$$

We take $K = \mathbb{Q}_p$ and use the case $k = 1$ of Theorem 2, which produces $\beta(1;\mathbb{Q}_p(T)) = 8$. We therefore conclude that

$$\beta(2;\mathbb{Q}_p,m) \le 2m + 8 \tag{7}$$

for all $m \geq 0$, which is easily shown to be best possible. Unfortunately it seems that we can get results of this quality only for the cases $r = 1$ and $r = 2$. We therefore ask:

**Open Question** *Is it true that* $\beta(3; \mathbb{Q}_p, m) = 2m + O(1)$ *uniformly for all* $m \geq 1$ *and all primes* $p$ *?*

Even the situation over $\mathbb{F}_p$ is unclear. One can use Amer's theorem to show that $\beta(2; \mathbb{F}_p, m) = 2m + 4$ for primes $p \geq 3$, and the result of Leep noted above similarly handles $p = 2$. However it appears to be unknown whether or not $\beta(3; \mathbb{F}_p, m) = 2m + O(1)$.

We can use (7) to advantage in our previous argument. From (4) we have $\beta(r; \mathbb{Q}_p) \leq \beta(2; \mathbb{Q}_p, \beta(r-2; \mathbb{Q}_p))$, so that (7) yields

$$\beta(r; \mathbb{Q}_p) \leq 2\beta(r-2; \mathbb{Q}_p) + 8.$$

In particular

$$\begin{aligned} \beta(3; \mathbb{Q}_p) &\leq 2\beta(1; \mathbb{Q}_p) + 8 = 8 + 8 = 16, \\ \beta(4; \mathbb{Q}_p) &\leq 2\beta(2; \mathbb{Q}_p) + 8 = 24, \\ \beta(5; \mathbb{Q}_p) &\leq 2\beta(3; \mathbb{Q}_p) + 8 \leq 40 \end{aligned} \tag{8}$$

using (8), and

$$\beta(6; \mathbb{Q}_p) \leq 2\beta(4; \mathbb{Q}_p) + 8 \leq 56.$$

One may then use (4) with $k = 1$ and (5) with $m = \beta(1; \mathbb{Q}_p) = 4$ to show that

$$\beta(7; \mathbb{Q}_p) \leq \beta(6; \mathbb{Q}_p) + 28 \leq 84.$$

From this point on the most efficient procedure is to use (4) with $k = 2$ and (5) with $m = \beta(2; \mathbb{Q}_p) = 8$, deducing that

$$\beta(r; \mathbb{Q}_p) \leq \begin{cases} 2r^2 - 16, & r \text{ even } \geq 6, \\ 2r^2 - 14, & r \text{ odd } \geq 7, \end{cases}$$

which improves on Martin's result (6) by 16. Thus the overall saving is not large, but is not insignificant for $r = 3$, for example.

In the simplest open case $r = 3$ our state of knowledge is therefore that

$$12 \leq \beta(3; \mathbb{Q}_p) \leq 16.$$

It is perhaps of interest to review the somewhat roundabout route to the upper bound here, since it combines results from both the lines of attack described here. The steps could be summarized as follows.

1. Theorem 1 handles systems of $r$ forms in at least $4r+1$ variables, over an extension $K$ of $\mathbb{Q}_p$, when $\#F_K \geq (2r)^r$.

2. Leep's argument for the proof of Theorem 2 shows that if

$$q(x_1,\ldots,x_9) \in \mathbb{Q}_p(T)[x_1,\ldots,x_9]$$

then one can find a zero over an extension $K(T)$, provided that $\#F_K$ is large enough.

3. On choosing a suitable odd degree extension, Springer's theorem produces a zero of $q(x_1,\ldots,x_9)$ over $\mathbb{Q}_p(T)$.

4. For suitable $n$ we may then find a large linear space of solutions for a form $q(x_1,\ldots,x_n)$ over $\mathbb{Q}_p(T)$.

5. Amer's theorem then shows that a pair of forms $q_i(x_1,\ldots,x_n)$ (for $i=1,2$) over $\mathbb{Q}_p$ also has a large linear space of solutions.

6. Combined with the estimate (4) this produces our bound for $\beta(3;\mathbb{Q}_p)$.

It certainly seems surprising that the proof goes via systems of large numbers of forms in steps 1 and 2. It would be interesting to know whether an argument based on minimal models could give a direct proof of a bound weaker than $\beta(3;\mathbb{Q}_p)=12$, but valid for all $p$.

# References


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